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FIXED POINT THEORY AND POSSIBILITIES FOR APPLICATION IN DIFFERENT FIELDS OF AN ECONOMY*

Teorija fiksne tačke i mogućnosti primene u različitim granama ekonomije

Abstract

This paper is the review article which presents the basic topics related to the fixed point theory. Two theorems regarding fixed point existence are presented: Brouwer's theorem and Kakutani's theorem. Both of them are widely used in different economic fields, especially for equilibrium price determination and the game theory. Possibilities for utilization of these theorems are vast, but this paper focuses on several heretofore known applications in the field of economic research. The primary goal of this paper is to describe the foundations of fixed point theory and outline some of the possible applications. More precisely, this is a starting point for future research regarding the determination of competitive relationship equilibrium in different markets.

Key words: *fixed point theory, equilibrium, Brouwer's theorem, Kakutani's theorem*

Sažetak

Ovo je pregledni članak u kome su navedeni osnovni pojmovi teorije fiksne tačke. Prezentovane su dve teoreme o postojanju fiksne tačke: Brauerova i Kakutanijeva. Ove teoreme su našle široku primenu u različitim granama ekonomije, pre svega u određivanju ravnotežne cene, kao i u teoriji igara. Naravno, mogućnosti primena su velike, tako da se u radu navodi jedan kratki segment dosadašnjih primena u ekonomiji. Cilj rada je da se prikažu osnovni pojmovi vezani za teoriju fiksne tačke i da se navedu neke moguće primene. Ovaj rad praktično predstavlja osnovu za buduće istraživanje koje bi se odnosilo na određivanje ravnotežnog konkurentskog odnosa na različitim tržištima.

Ključne reči: *teorija fiksne tačke, ravnotežna tačka, Brauerova teorema, Kakutanijeva teorema*

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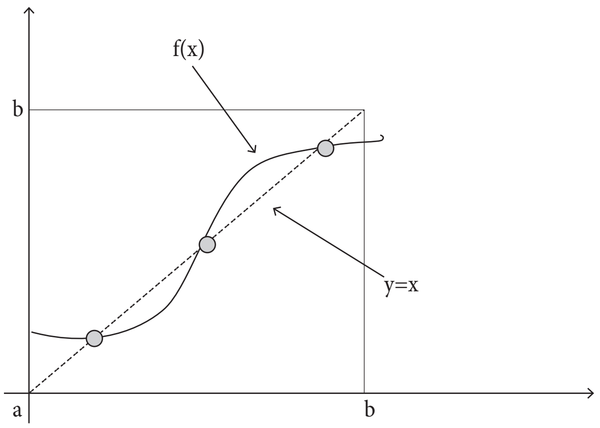
Introduction

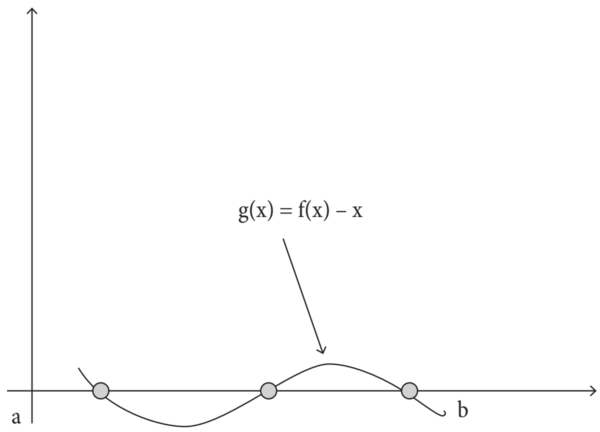
Fixed point theory examines the existence of the point x belonging to the domain of function f for which stands that $f(x) = x$, i.e. function values are equivalent to identical function mapping. In Figure 1 three intersections of function $f(x)$ and function $y = x$ represent the fixed points. A more subtle analysis would lead to the conclusion that a marginal change in the $f(x)$ function causes additional fixed points to emerge.

Figure 1: Three fixed points of the function f



If a certain function g is presented as $g(x) = f(x) - x$, than the solution to the equation $g(x) = 0$ is the fixed point of the function f (see Figure 2).

Figure 2: Solution to the equation g(x) = 0 is the fixed point of function f



Fixed point theory is applied in different scientific fields. In mathematics, it is used for solving different equations, creating approximations and simulations, in game theory, etc. In the field of economics, it is often used in the process of determining the coincidence point of supply and demand functions. Actually, fixed points (i.e. equilibriums) are at the core of many generic economic models. This theory enhanced the understanding of many other problems inherent to economic models such as comparative statics, robustness to marginal changes and equilibrium stability as well as equilibrium calculation.

One of the pioneer theorems regarding the fixed point is the *Brouwer's theorem* (refer to [2] for more details). The proof of the Brouwer's fixed point theorem is one of the most important results in the history of topology because it initiated a substantial number of generalizations and broadened its effects to different fields of mathematics and other scientific disciplines. *John von Neumann* [12] used it first to prove the existence of a "minimax" solution to two-agent games. He used a generalization of Brouwer's theorem again (in 1937) to prove existence of a balanced growth equilibrium for his expanding economy (refer to [13] for more details). This generalization had been simplified by *Kakutani* (1941). Fixed point theorems (*Kakutani's theorem* especially) made it possible to prove the crucial theorems in *Nash* [8], [9] for the case of non-cooperative games as well as *Arrow* and *Debreu* [1] on general equilibrium theory. Brouwer's theorem was used in the papers [5], [10], [11] and many others.

In this paper, the basic results of fixed point theory valuable to the economic researches are reviewed. The primary goal of this paper is to present Brouwer's and Kakutani's theorems in order to analyze potential applications in the field of economic research.

Brouwer's and Kakutani's theorems

Brouwer's and Kakutani's theorems are presented in this section[1].

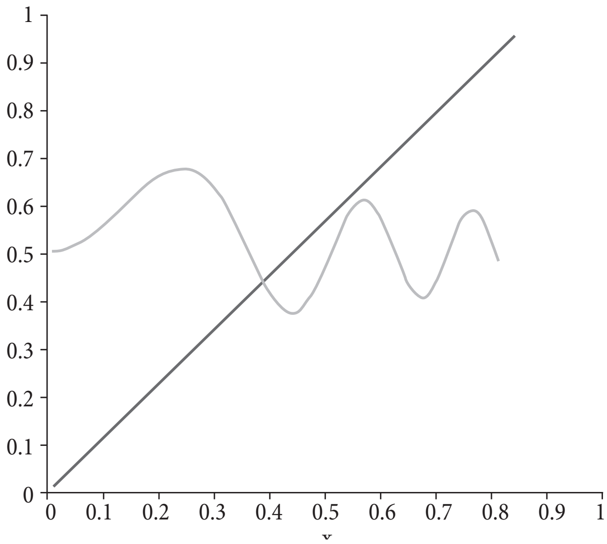
1 The following labels should be introduced in order to make mentioned theorems more understandable. Let X be a set, and let T be a family of subsets to the set X for which the following stands:
- The empty set and X belong to T;
- Any union of elements from T is an element of T;
- The intersection of any finite number of sets from T belong to T.

T is regarded as topology on X and that (X, T) is a topological space. A set from T is called an open set. A set which is a complement to the set from T is called a closed set. A set is convex if for every two points x, y from that set a point $tx+(1-t)y$ also belongs to this set (whereas t is within interval [0,1]). A set is compact if for each sequence from this set there is a subsequence that converges to some point from the set. Besides that, a set is compact if it is closed and bounded.

Brouwer's theorem [2]. Let $X \subseteq \mathbf{R}^n$ be nonempty, compact, and convex, and let $f: X \to X$ be continuous. Then f has a fixed point.

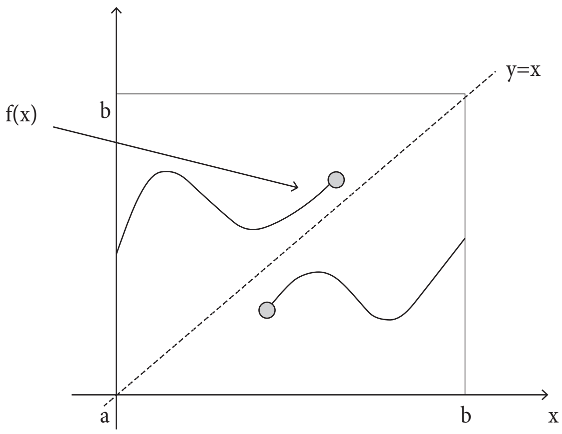
Application of this theorem makes it possible to conclude, for example, that continuous function that maps the interval [0,1] to [0,1] has a fixed point (see Figure 3).

Figure 3: Fixed point of the continuous function within interval [0,1]



In Figure 4 is shown a function which is not continuous so within interval [*a*,*b*] it does not have a fixed point.

Figure 4: Fixed point does not exist for a function which is not continuous on [a,b]



It is common for economic models, particularly those in the field of game theory to account for settings in which agents have more than one rational choice at their disposal. The first generalization of Brouwer's theory emphasizes on this. Let us introduce the following labels (see [6]):

- If X and Y are sets, a correspondence $F: X \to Y$ is a function from X to the nonempty subsets of Y.
- If Y is a topological space, F is compact valued if, for all $x \in X$, $F(x)$ is compact.
- If Y is a subset of a vector space, then F is convex valued if each $F(x)$ is convex.

In order to apply Brouwer's theorem to correspondences it is necessary to define the continuity of correspondences (see [6]):

- If X and Y are topological spaces, a correspondence $F: X \to Y$ is upper semicontinuous if it is compact valued and, for each $x_0 \in X$, and each neighborhood $V \subset Y$ of $F(x_0)$, there is a neighborhood $U \subset X$ of x_0 such that $F(x) \subset V$ for all is $x \in U$.
- Fixed point of a correspondence $F: X \in X$ is a point x^* for which holds $x^* \in F(x^*)$.

The version of fixed point theorem most frequently used in economic analysis had been proven by *Kakutani* [3].

Kakutani's theorem [3]: If $X \subset \mathbf{R}^n$ is nonempty, compact, and convex, and F: $X \to X$ is an upper semicontinuous convex valued correspondence, then F has a fixed point.

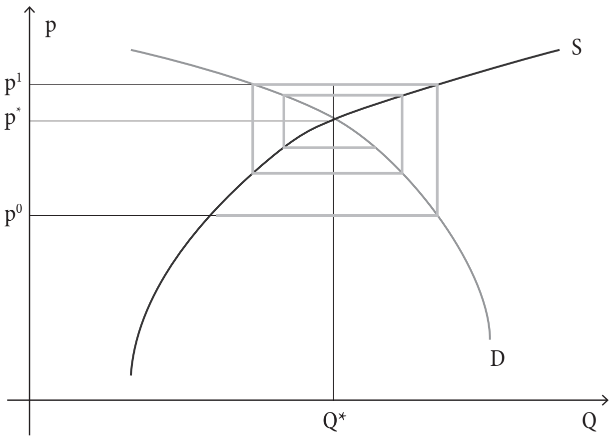
The economic application of Brouwer's and Kakutani's theorems

Fixed point theorems are most frequently used for proving that at least one equilibrium exists in an economic or game theory model. Equilibrium is the vector of endogenous model variables when all agents are presumed to act rationally, through utility maximization, and when an individual agent regards all other endogenous variables *ceteris paribus*.

Application 1. Let P be the price and Q the quantity. Let $P=D(Q)$ be the demand function and $P=S(Q)$ supply function. If supply is equal to demand then there exists market equilibrium, presented with equilibrium [Q^*, P^*] (Q^* being the equilibrium quantity and P^* being the equilibrium price, see Figure 5).

Market price differs from equilibrium price due to effects of competition. That is why a market is regarded as stable when price converges to equilibrium price.

Figure 5: The equilibrium



Let P_{min} be the lowest price for a commodity in the given market. Let P_{max} be the highest price at which a commodity can be sold in the given market. Observe the following function:

$$f:[P_{min}, P_{max}] \rightarrow [P_{min}, P_{max}]$$

defined with:

$$f(P) = D(S^{-1}(P_0)),$$

Function f is adequately defined for a given price domain because the monotony of demand and supply function allows for the existence of adequate inverse functions. It is straightforward to prove that function is continuous. Since domain $[P_{min}, P_{max}]$ is compact and convex, it can be concluded that fixed point (price) exists on the basis of Brouwer's theorem.

Let us describe the algorithm used in order to determine equilibrium price. Let P_0 be the market price which is lower than equilibrium price, i.e. $P_0 < P^*$. Let Q^0_D and Q^0_S be the demand quantity and supply quantity, respectively for price P_0. The following is true then:

$$Q^0_D = D^{-1}(P_0) \text{ and } Q^0_S = S^{-1}(P_0).$$

Given that D is monotonic decreasing function and D is monotonic increasing function then inverse functions D^{-1} and S^{-1} exist. If producers increase the price to P_1 (for a demanded quantity) then the following is true:

$$P_1 = D(Q^0_S) = D(S^{-1}(P_0)),$$

and $P_1 > P^*$. The following stands for corresponding demand and supply quantities Q^1_D and Q^1:

$$Q^1_D > Q^1_S,$$

which leads to deviation of $|Q^1_S - Q^1_D|$. If producers decrease the price to P_2 so that:

$$P_2 = D(Q^1_S) = D(S^{-1}(P_1)).$$

If we repeat this algorithm, we get the sequence of the prices $P_0, P_1, P_2, \dots, P_k, \dots$ for which:

$$P_k = D(S^{-1}(P_{k-1})), k = 1,2,\dots$$

According to that the sequence of the prices (P_k) converges to the equilibrium price P^*. This is presented in Figure 5. *Meznik* [7] has also considered this application.

Application 2 (Nash equilibrium). Let N be a fixed finite set, which is called "set of players (participants)". Each player is labeled with index i.

Normal-form game is an ordered triple, in which for every $i \in N$, S_i is non-empty sets, and u_i is functions $u_i : \prod_{i\in N} S_i \rightarrow \mathbf{R}$. We will regard S_i as a set of strategies, and i as a user's gain (utility) function ($i \in N$). If we denote $S_N = \prod_{i\in N} S_i$, then every $s \in S_N$ is the *outcome* (strategic profile) in the game Γ. Player i chooses strategy $s_i \in S_i$. When all players choose their strategies, then the outcome of game s and gain for every player $i - u_i(s)$.

From the aforementioned the single *normal-form game* is defined when the following three elements are defined:

1) set of game participants,
2) set of strategies for each player,
3) gain function for each player.

Firstly, several useful notations will be introduced. Let $s = (s_1, s_2, \dots, s_n)$ be a strategic profile. Then:

1) $s_{-i} = (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$
2) $(s_{-i}, s^*_i) = (s_1, s_2, \dots, s_{i-1}, s^*_i, s_{i+1}, \dots, s_n)$

Nash equilibrium is the strategic profile $s^* \in S$ in which for every $i \in N$ stands that $u_i(s^*_{-i}, s^*_i) \geq u_i(s^*_{-i}, s_i)$ for $s_i \in S_i$.

Nash theorem [8], [9]**.** If strategic sets of each player are non-empty, convex and compact and their utility functions are continuous and quasiconcave for s_{-i} then Nash equilibrium exists for a normal-form game.

The proof of this theorem is implied by Kakutani's theorem since the best answer function is defined with $b_i(s_{-i}) = \arg\max \{u_i(s_i, s_{-i}) | s_i \in S_i\}$ and $b(s) = \prod_{i=1}^{n} b_i(s_{-i})$. Function b is well defined on the basis of *Weierstrass theorem*. It should be noticed that if $s^* \in b(s^*)$ then $s^*_i \in b(s^*_{-i})$ for every $i \in N$, which leads to the conclusion that s^* is Nash equilibrium.

Application 3 (*Cournot oligopoly,* see [4]). *Cournot oligopoly model* is the model for which holds the next assumptions:

- there are n firms;

- a firm i produces commodity i for $i \in \{1,2,\ldots,n\}$($q_i \geq 0$ is the quantity of commodity and p_i is the price);
- all goods (commodities) are perfectly divisible;
- the goal of each firm is to choose an amount of product that maximizes its own profit given the production levels chosen by other firms.

Let $q_{-i} = (q_1,\ldots, q_{i-1}, q_{i+1},\ldots,q_n)$ be a vector of quantities produced by the other firms. We can assume that:

$$p_i = P_i(q_i, q_{i-1}) = a_i - b_i q_i + \sum_{j \neq i} d_{ij} q_j,\ i = 1,\ldots, n$$

i.e. price p_i is decreasing in its own quantity q_i and, due to complementarities between the commodities, is assumed to be increasing in the quantities $q_j, j \neq i$, of the other firms (parameters a_i, b_i, d_{ij} are positive).

Each firm $i \in \{1,2,\ldots,n\}$ has a linear cost function:

$$C_i(q_i) = c_i q_i$$

with $a_i > c_i > 0$. The profit π_i of firm $i \in \{1,2,\ldots,n\}$ is:

$$\pi_i(q_i, q_{-i}) = q_i P_i(q_i, q_{-i}) - c_i q_i.$$

A tuple $(q^*_1,\ldots, q^*_n) \in \mathbf{R}^n_+$ is a *Cournot-Nash equilibrium* if for every firm $i \in \{1,2,\ldots,n\}$ holds:

$$\pi_i(q^*_i, q^*_{-i}) \geq \pi_i(q_i, q^*_{-i})$$

for all $q_i \in \mathbf{R}_+$. This equilibrium exists if $2b_i > \sum_{j \neq i} d_{ij}$ for every firm $i \in \{1,2,\ldots,n\}$.

Discrete Cournot-Nash equilibrium is analyzed when the assumption that all commodities are perfectly divisible is not satisfied. Some commodities, like cars, machines, etc. are produced and sold in integer quantities. Also many divisible goods are sold in discrete quantities, like barrels of oil or grain.

A tuple $(q^*_1,\ldots, q^*_n) \in \mathbf{Z}^n_+$ is a discrete Cournot-Nash equilibrium if for every firm $i \in \{1,2,\ldots,n\}$ holds:

$$\pi_i(q^*_i, q^*_{-i}) \geq \pi_i(q_i, q^*_{-i})$$

for all $q_i \in \mathbf{Z}_+$. That is, given the integer quantities chosen by other firms, each firm chooses an integer quantity that yields a profit which is at least as high as any other integer quantity could give.

A firm $i \in \{1,2,\ldots,n\}$ can maximize its profit $\pi_i(q_i, q_{-i})$ if its optimal integer quantity is given by the reaction function:

$$r_i(q_{-i}) = \left[\frac{a_i - c_i}{2b_i} + \sum_{j \neq i} \frac{d_{ij}}{2b_i} q_j\right]$$

The symbol $[x]$ denotes the greatest nearest integer to x and for $i \in \{1,2,\ldots,n\}$ holds $r_i(q_{-i}) \geq 0$ for every $q \in \mathbf{Z}^n_+$ (because $a_i > c_i > 0$).

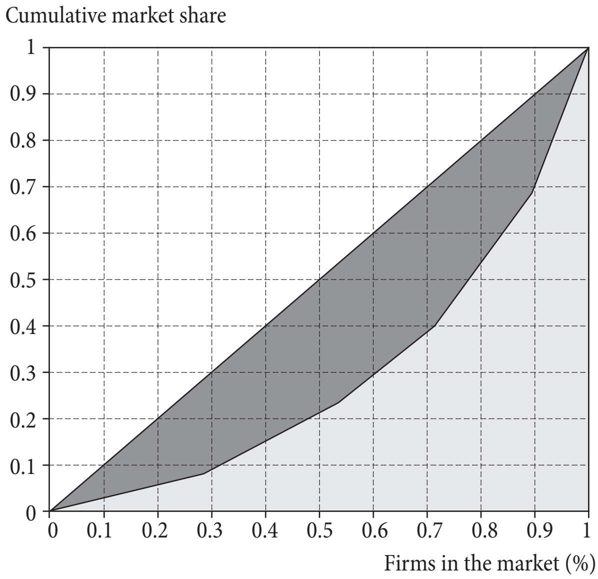
Define the function $f: \mathbf{Z}^n_+ \to \mathbf{Z}^n$ by

$$f_i(q_i, q_{-i}) = r_i(q_{-i}) - q_i,\ i = 1,\ldots, n.$$

A discrete zero point of f is a discrete Cournot-Nash equilibrium. Brouwer's fixed point theorem can show that function f will have a discrete zero point if $2b_i > \sum_{j \neq i} d_{ij}$, $i = 1,\ldots, n$. This means that Cournot oligopoly model with complementary commodities will have a discrete Cournot-Nash equilibrium when $2b_i > \sum_{j \neq i} d_{ij}$, $i = 1,\ldots, n$.

*Application 4 (Measuring market concentration)***.** Concentration curve is a popular tool for visualizing market concentration and perceiving the market strength inequality. The steps in order to draw concentration curve are the determination of competitor ranking in terms of market share (smallest to largest), cumulative competitor market share and joining the dots points created in the process. Newly drawn concentration curve is then compared to the curve representing equal market shares (45°line) in the hypothetical perfect competition setting (Figure 6).

Figure 6: Concentration curve



Actually, concentration curve is a graph of continuous function f which maps interval [0,1] to [0,1]. Since assumptions of Brouwer's theorem are satisfied, a fixed point for this mapping exists. More precisely, this fixed point is not unique since from Figure 6 can be observed that both points 0 and 1 are the fixed points. In case of the perfectly equally distributed market strength, Gini coefficient would be equal to zero since competition curve would be identical to the curve representing equal market shares. Since for the whole domain $f(x) = x$ would be true, and an indefinite number of fixed points for this function would exist. It should be, however, taken into account that such an extreme situation is empirically rare.

Similarly to concentration curve Lorenz curve depicts the level of household income inequality. When income discrepancy is large then the curve is substantially remote from the 45^o line. The less the inequality the more will curve converge to 45^o line. In the case of perfect equality, as with concentration curve, perfect equality leads to the Lorenz curve with indefinite number of fixed point. It should be, however, outlined that such a case is empirically rare.

*Application 5 (Competitive dynamics within industry)***.** Primary goal of this segment is to define the framework for further research in prospective papers. Let us first assume that there is a market structure with characteristics of duopoly in ice cream wholesale industry in Serbia (Frikom and Nestle Ice Cream). Frikom dominates the market with a market share of 82% while the main follower is Nestle with market share of 12%. After market share distribution and dynamics leading to competitive balance are assessed it is observed that these factors remained stable during the last 4 years, which leads to the conclusion that certain form of competitive equilibrium is established, i.e. that a fixed point exists.

Analyzing the history of competitive dynamics for these two market participants provides interesting conclusion since the industry went from one competitive equilibrium to another. Nestle Ice Cream was dominant market participant with almost 60% market in 2000. At the same time, Frikom had a market share of 29% and was on the brink of bankruptcy mainly because of serious liquidity issues. The turning point was the acquisition of Frikom by Croatian company Agrokor. Agrokor invested aggressively in all elements of business (R&D, marketing, employee education, transportation, equipment, refrigerating systems, etc.). Distribution model was also changed from distributor oriented to capillary model. Aggressive investment, fresh know-how and brave managerial decisions led to a steep market share increase that peaked in 2009 at 82%. Despite intensive competitive efforts by Nestle in the previous 4 years (including organizational redesign, changes in the management team, improvements of distribution model, aggressive rebate-based discount strategy, among others) market share equilibrium remained nearly completely intact. The intention of future research efforts and papers would be to explain outlined transition of competitive equilibrium by using theorems explained in the previous sections of this paper. Final result of research would be the generalization of findings to other industries.

Conclusion

This paper is the starting point for further research on the fixed point theory application in economics. In order to clarify the potential scope of utilization for economic research purposes elementary topics of fixed point theory are hereby introduced. Brouwer's and Kakutani's theorems reviewed in this paper are the basis for further analysis and assessment of equilibrium. Although its application is a great challenge, this theory draws attention of mathematicians all around the world. Authors of this paper intend to apply this theory to the research of topics such as market concentration and competition as well as the determination of equilibrium states.

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